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LETTER TO THE EDITOR

Rupture in a chain of fibre bundles

B Kahng†

Centre for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA

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Abstract. We consider the electrical breakdown properties of a series of L bundles, each containing w fibre elements. Each fibre is supposed to be a resistor having resistance r_a with probability p, and having resistance r_b with probability $1-p$, provided that the power dissipated over the fibre is less than a threshold value. However, the fibre burns out and changes into an insulator for higher powers. In this model, two fundamentally different breaking behaviours appear depending on the parameters w, p, and $h \equiv r_b/r_a$. For one case, breaking occurs only on the weakest bundle. For the other case, breaking propagates across bundles. In the latter case, the configuration average external voltage has to be increased by $\sim w^{-1/2}\exp(-w)$ to reach the final breakdown after the first breaking. We also find that the total number of bonds broken becomes a maximum when the ratio of the two resistances is equal to 4 for $w \to \infty$. In addition, we discuss the application of this model to the problem of 'a stand-by system without repair'.

Recently electrical breakdown problems [1] have drawn much attention as an analogy for mechanical fracture in disordered media. For example, a resistor-fuse network with a constant breaking strength in percolation disordered systems [1-3] or with random breaking strengths in homogeneous systems [4] were introduced to study fracture phenomena in composite systems. In these systems, the breaking of individual bonds is caused by the coherent effects, screenings or enhancements, induced by many defects. Even in quasi-one-dimensional fibrous structures, the coherent effects between many flaws play a crucial role in the breaking process. This strongly correlated phenomena makes it difficult to analyse macroscopic breaking behaviour. Phoenix *et al* [5-6] have studied rupture in the fibrous bundles for many years, for example breakdown of the fibre bundles when the breaking strength of each fibre is not identical. In this case, breakdown of the entire system is not always caused by breaking on a single bundle, but occurs after crack propagation across many bundles. Due to this complicated mechanism, analytic understanding associated with rupture in the fibrous bundles still remains unclear. Recently, Sornette [7] also considered failure of the strings in a series of bundles, in which microscopic understanding of the breaking process is absent.

In this work, we introduce and consider an electrical breakdown model of a quasi-one-dimensional structure composed of two different types of resistors in order to study the breaking process in fibrous bundles. More specifically, the system consists of a series of L bundles (blobs), each of which contains w strands (bonds) as shown in figure 1. The strands in a bundle are composed of two different types of resistor having resistances r_a and r_b ($r_a < r_b$), with probability p and $1-p$ respectively. Each

† Present address: Department of Chemistry, University of California, Berkeley, CA 94720, USA

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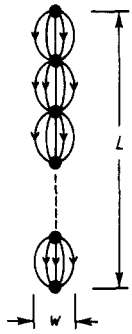

Figure 1. Schematic picture of the two-component bubble model. The bonds with the resistance r_a are indicated by arrows.

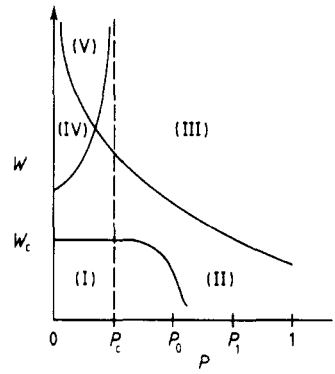

Figure 2. Phase diagram in the parameter space by p and w. Regions (I)-(V) indicate different regimes defined in the text.

resistor burns out and turns into an insulator if the power dissipated on the resistor is beyond the critical value, say 1. We assume that when the number of the weakest bonds is more than one, all of them burn out simultaneously. After the first breaking occurs, we also assume that there is sufficient time for the modified network, in which the hottest bond has been removed, to reach a new equilibrium voltage state before further fuse burning takes place. Next, we recalculate the new equilibrium voltage distribution. Now a new weakest bond may be identified, and then broken. This process is repeated until the network is divided into two pieces. In the breaking process, the system is maintained by constant external current I. This model is a simple description of a series of composite fibre bundles in the sense that bonds in the same bundle are indistinguishable in their relative positions, and breaking strengths of individual fibres are assumed to be uniform. In spite of the simplications, we can still observe interesting features of breaking propagation.

A similar chain structure, the bubble model, has been introduced previously [8]. The bubble model is the $r_b \to \infty$ limiting case, of the model we consider here. The bubble model was introduced to describe percolation cluster by choosing the length-width relation $L \sim e^w$. With this relation, a non-trivial percolation threshold can exist in a finite reigon of probability p. We will use this relation in this problem. Using the bubble model, we studied the maximum voltage drop in a percolation-type random resistor network. When the correlation length ξ in percolation clusters is much less than the system size L, the voltage drops across horizontal bonds are very small. Hence we may neglect horizontal bonds when we consider a problem associated with positive higher moments of the voltage drop distribution, like the problem of the random fuse model [1]. Thus squeezing horizontal bonds, which generates a quasi-one-dimensional structure like the bubble model, does not change the essence of the physics and makes the problem soluble. However, it is not obvious that the two-component bubble model we consider represents the random resistor network [9] which consists of two different types of resistors with resistances r_a and r_b with probability p and $1-p$ respectively in a $L \times L$ system. However, the two-component bubble model is useful in its own right for the study of rupture phenomena in fibre bundles.

We obtain two microscopically different breaking processes depending on the probability p, the ratio of two resistances r_b/r_a, and the system size w. For one case,

breaking occurs only on a single bundle, and for the other case breaking occurs on many bundles in a catastrophic way. In both cases we obtain breaking strengths and breaking probability distributions analytically. In the latter case, we have to increase the external voltage by $w^{-1/2}\exp(-w)$ to reach the final breakdown after the first breaking occurs. This increment turns out to be a more fundamental quantity in the sense that the breaking probability distribution function is expressed in terms of the increment. The failure distribution follows the Weibull distribution. The Weibull constant is found to be non-universal and depends on the probability p and the two resistances r_a and r_b. Finally we argue that this model can be applied to the problem of 'stand-by systems without repair' [10].

Let us begin by considering a single blob (bundle) composed of k bonds with resistance r_a and $w-k$ bonds with resistance r_b. Hereafter we shall call this kind of blob a 'k-blob', and call a bond with resistance r_a, (r_b) an 'A-type' (B-type) bond. In this structure, if the blob is biased by the current I, the currents flowing through an A-type bond and a B-type bond, $i_a(k)$, $i_b(k)$, are

$$i_a(k)=\begin{cases}\dfrac{r_b I}{kr_b+(w-k)r_a} & \text{for } k\neq 0\\ 0 & \text{for } k=0\end{cases} \tag{1}$$

$$i_b(k)=\begin{cases}\dfrac{r_a I}{kr_b+(w-k)r_a} & \text{for } k\neq w\\ 0 & \text{for } k=w\end{cases} \tag{2}$$

and the power dissipated over an A-type bond and a B-type bond is given by

$$Q_a(k)=\begin{cases}\dfrac{r_b^2 r_a I^2}{[k(r_b-r_a)+wr_a]^2} & \text{for } k\neq 0\\ 0 & \text{for } k=0\end{cases} \tag{3}$$

$$Q_b(k)=\begin{cases}\dfrac{r_a^2 r_b I^2}{[k(r_b-r_a)+wr_a]^2} & \text{for } k\neq w\\ 0 & \text{for } k=w.\end{cases} \tag{4}$$

For the case $k\neq 0$ or $k\neq w$, the ratio of the powers dissipated in each type of bond is simply $Q_a(k)/Q_b(k)=r_b/r_a>1$, which implies that A-type bonds do break earlier than B-type bonds. Moreover, because $Q_a(k)>Q_a(k+1)$, A-type bonds in k-blobs break earlier than A-type bonds in $(k+1)$-blobs. But if a blob composed only of B-type bonds exists, we have to compare $Q_a(1)$ with $Q_b(0)$ to determine the hottest bond. After simple calculations, we obtain $Q_a(1)>Q_b(0)$ when $w>w_c$ such that

$$w_c=\frac{(r_b-r_a)}{\sqrt{r_a r_b}-r_a}=\left(\frac{r_b}{r_a}\right)^{1/2}+1. \tag{5}$$

Thus if $w<w_c$ then the hottest bond is a bond in the blob composed only of B-type bonds. But if $w>w_c$ then the hottest bond is an A-type bond in the 1-blob. For the former case, the initial breaking leads to breakdown of the system, but for the latter case the initial voltage required to break the hottest bond is not sufficient to lead to breakdown of the entire system. Thus we need to increase the external voltage to reach the failure of the system.

However the 0-blob does not have a finite probability of existing in the entire region of probability p. As we studied in [8], the probabilities for a 0-blob and for a 1-blob to exist are given respectively by

$$P_0(p; L) = 1-(1-q^w)^L \tag{6a}$$

$$P_1(p; L) = 1-(1-wpq^{w-1})^L. \tag{6b}$$

These probabilities show sharply decreasing behaviour in a finite region of probability as shown in figure 4 of [8]. Thus when $p < p_0$, depending on the ratio of $Q_b(0)$ and $Q_a(1)$, the initial breaking takes place either on A-type bonds or on B-type bonds. If the breaking is initiated by the B-type bonds, then the initial breaking immediately leads to breakdown of the system. But the breaking initiated by A-type bonds does not lead to the system breakdown. However when $p > p_0$, breaking begins with A-type bonds in the blob having the smallest number of A-type bonds.

More precisely, we first consider the trivial case that breaking only occurs on B-type bonds, which occurs for $w < w_c$ and $p < p_0$ (region (I) in figure 2). The power dissipated on a B-type bond is

$$Q_b(0) = \left(\frac{r_b}{w^2}\right) I^2 = \left(\frac{r_b}{w^2}\right)\left(\frac{V}{R}\right)^2 \tag{7}$$

where V is the external voltage and R is the total resistance of the system. Since the initial breaking occurs when the power $Q_b(0) = 1$, the breaking voltage is $V_b = (w/\sqrt{r_b})R$. The average breaking voltage over configurations is $\langle V_b\rangle = (w/\sqrt{r_b})\langle R\rangle$, and the average resistance is found to be

$$\langle R\rangle = L\sum_{k=0}^{w} \frac{r_a r_b}{wr_a + wpk(r_b - r_a)} \binom{w}{k} p^w q^{w-k}. \tag{8}$$

For large w, we apply the central limit theorem to evaluate (8), and obtain

$$\begin{aligned}\langle R\rangle &\sim \frac{L}{\sqrt{2\pi}} \int_a^b \frac{r_a r_b}{wr_a + (r_b - r_a)wp + t\sqrt{wpq}} \exp(-\tfrac{1}{2}t^2)\,\mathrm{d}t \\ &\sim \frac{L}{w}\frac{r_a r_b}{r_a + p(r_b - r_a)}\end{aligned} \tag{9}$$

where we used the relation $k = wp + t\sqrt{wpq}$, thus $a = -\sqrt{wp/q}$, and $b = \sqrt{wq/p}$. From (4), we obtain

$$\langle v_b\rangle \equiv \frac{\langle V_b\rangle}{L} \sim \frac{r_a\sqrt{r_b}}{r_a + p(r_b - r_a)} \equiv v_0(p, h) \tag{10}$$

where $v_0(p, h)$ means a function of p, h, but not of the system size w.

Next consider the case when breaking is triggered by burning of the A-type bonds. In order to determine which bonds should break after the first burning, we think of breaking A-type bonds in a k-blob. After A-type bonds break, the power dissipated on B-type bonds in the k-blob is

$$Q_b'(k) = r_b I^2/(w-k)^2. \tag{11}$$

If this power $Q_b'(k)$ is larger than the power $Q_a(k+1)$ on A-type bonds in $(k+1)$-blobs, then the next breaking does not propagate to other blobs, but occurs on B-type bonds in k-blobs. However, when $Q_b'(k) < Q_a(k+1)$, breaking propagates to the $(k+1)$-blobs. In this case, after breaking is initiated on k_0-blobs, where k_0-blobs are the

blobs with the narrowest width of A-type bonds in the system, breaking propagates to (k_0+1)-blobs, and (k_0+2)-blobs, so on. Here the narrowest width k_0 depends on probability p. This propagation will continue up to k_c-blobs which satisfy $Q_b'(k_c) \geqslant Q_a(k_c+1)$. From (3) and (11), we obtain k_c to be

$$k_c = \frac{(\sqrt{r_a r_b} - r_a)w - (r_b - r_a)}{r_b - r_a + \sqrt{r_a r_b}}. \tag{12}$$

Specially if k_0-blob satisfies the condition, $Q_b'(k_0) > Q_a(k_0'+1)$, then the breaking occurs only in k_0-blobs. This implies the case $k_0 > k_c$, which appears in region (II) in figure 2.

After the A-type bonds in k_c-blobs break, the power disspated on a B-type bond in the same blob is

$$Q_b'(k_c) = \frac{r_b I^2}{(w-k_c)^2} = \frac{r_b}{(w-k_c)^2}\left(\frac{V}{R_f}\right)^2 \tag{13}$$

where R_f is the resistance of the system after breaking of A-type bonds up to k_c-blobs. If this power is larger than 1, the B-type bonds in k_c-blobs break, and the system reaches breakdown. Hence the average breaking voltage is

$$\langle V_b \rangle = \langle R_f \rangle \frac{w - k_c}{\sqrt{r_b}}. \tag{14}$$

In order to evaluate the breaking voltage $\langle V_b \rangle$, we need the final resistance $\langle R_f \rangle$, which is evaluated as

$$\langle R_f \rangle = \langle R_i \rangle + L \sum_{k=k_0}^{k_c} \left(\frac{r_b}{w-k} - \frac{r_a r_b}{k(r_b - r_a) + w r_a} \right) \binom{w}{k} p^k q^{w-k}. \tag{15}$$

Again for large w, we use the central limit theorem to evaluate (15), and obtain that

$$\langle R_f \rangle - \langle R_i \rangle = \frac{L}{\sqrt{2\pi}} \int_a^b \left(\frac{r_b}{wq - t\sqrt{wpq}} - \frac{r_a r_b}{w r_a + (r_b - r_a)wp + t\sqrt{wpq}} \right) \exp(-\tfrac{1}{2}t^2)\, dt \tag{16}$$

where $a = (k_0 - wp)/\sqrt{wpq}$, and $b = (k_c - wp)/\sqrt{wpq}$. We change the variables, t, a, and b to dimensionless variables t', a', and b', which are defined by $t = \sqrt{w}t'$, $a = \sqrt{w}a'$ and $b = \sqrt{w}b'$. Hence, if we drop the prime notation, (16) can be rewritten as

$$\langle R_f \rangle - \langle R_i \rangle = \frac{L}{\sqrt{2\pi w}} \int_a^b \left(\frac{r_b}{q - t\sqrt{pq}} - \frac{r_a r_b}{r_a + (r_b - r_a)p + t\sqrt{pq}} \right) \exp(-\tfrac{1}{2}wt^2)\, dt \tag{16'}$$

where $a = (k_0/w - p)/\sqrt{pq}$ and $b = (k_c/w - p)/\sqrt{pq}$.

Equation (16') is a Laplace integral [11], and the asymptotic behaviour of the integration depends on the signs of a and b. The final result is so complicated that we present only the size-dependent behaviour as follows. (i) When $a < 0$ and $b < 0$ (region (III) in figure 2), or $a > 0$ and $b > 0$ (region(IV)), we use the method of integration by parts, and obtain that

$$\langle R_f \rangle - \langle R_i \rangle \sim L w^{-3/2} \exp(-w) \qquad \langle R_f \rangle \sim L w^{-1} + L w^{-3/2} \exp(-w) \tag{17}$$

with inappropriate constants set to 1. (ii) When $a < 0$ and $b > 0$ (region (V)), we use the steepest descent method, and obtain that

$$\langle R_f \rangle - \langle R_i \rangle \sim L w^{-1}. \tag{18}$$

Since $\langle R_i \rangle \sim Lw^{-1}$, $\langle R_f \rangle \sim Lw^{-1}$. Hence, using (14), we obtain the average breaking voltage to be

$$\langle v_b \rangle \equiv \frac{\langle V_b \rangle}{L} \sim \begin{cases} v_0(h, p) + w^{-1/2} \exp(-w) & \text{for case (i)} \\ v_0(h, p) & \text{for case (ii)} \end{cases} \tag{19}$$

where $v_0(p, h)$, given in (10), means a function of p, and h, but not of the size w. Hence the average breaking voltage shows the breaking propagation mechanism to be exponentially dependent on the system size for a broad range of values of the probability p, the regions (III) and (IV). However, even in case (ii), we do not exclude the possibility of having a correction term of $\mathcal{O}(e^{-w})$.

Since we know the size-dependent behaviour of the breaking voltage explicitly for case (i), we focus on case (i) to consider the breaking probability distribution. When the system is biased by the breaking voltage v_b, the system would not break if it is in a configuration such that the narrowest width of A-type bonds is larger than k_c. This configuration exists with the probability $G(k_c)$:

$$G(k_c) = \left(1 - \sum_{k=0}^{k_c} \binom{w}{k} p^k q^{w-k}\right)^L. \tag{20}$$

Hence the probability that the system breaks is $F(k_c) = 1 - G(k_c)$. Again using the central limit theorem, we evaluate $\Sigma_0^{k_c} \binom{w}{k} p^k q^{w-k}$ and find $G(k_c)$ to be

$$G(k_c) \sim (1 - mw^{-1/2} \exp(-lw))^L \tag{21}$$

where m and l are constants independent of w, but they depend on the probability p and the resistances r_a, r_b. Using the breaking voltage, (19), we derive the breaking probability $F(v_b)$:

$$F(v_b) \sim 1 - \exp(-AL(v_b - v_0)^l) \tag{22}$$

where A is a function of w, but not of v_b. Hence it turns out that the breaking probability is of the Weibull distribution with the argument $v_b - v_0$. The quantity $v_b - v_0$ represents the increment of the external voltage through the breaking propagation. Hence we may say that the increment is a more fundamental quantity in the distribution function. We test the failure distribution by numerical simulation at $p = 0.6$ with $w = 10$, for different resistance ratios, $r_a = 3$, $r_b = 6$ and $r_a = 3$, $r_b = 9$ in figure 3. Both cases obey the Weibull distribution, but with different slopes, which confirms the theoretical prediction.

At this stage, it would be interesting to apply this model to the problem of a stand-by system without repair or a service system with waiting lines [10]. If we regard B-type bonds as supplementary devices, then after A-type bonds break, B-type bonds take over the operation, because burnings occur only on A-type bonds up to k_c-blobs. An interesting inquiry associated with the stand-by systems may be to determine when the stand-by system can play its role most effectively. That reduces to the question deciding what value of the ratio of the two resistances maximises number of bonds broken. Thus from (12), k_c is maximum when the ratio of two resistances is

$$h_c = \left(\frac{w + \sqrt{w^2 - w - 1}}{w + 1}\right)^2. \tag{23}$$

Hence when $w \to \infty$, the total number of fibres broken in the breaking process becomes maximum when the ratio h_c is ~ 4.

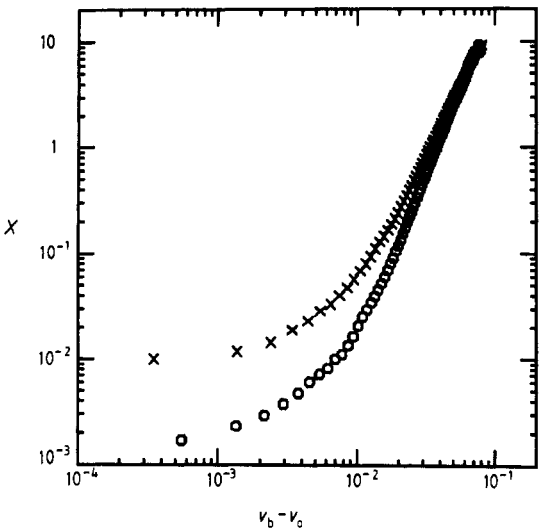

Figure 3. The distribution of breaking strengths $F(v_b)$. Plot of $X \equiv -\ln[1-F(v_b)]$ as a function of $v_b - v_0$ on a double logarithmic scale, for the cases $w = 10$, $p = 0.6$, $r_a = 3$, $r_b = 6$, (○) $r_a = 3$, $r_b = 9$, (×) for 10^4 configurations in both cases. We choose values $v_0 = 1.62$(○), $v_0 = 1.42$(×), obtained by attempting to fit to the Weibull form. The phenomenological values v_0 may in part cause the data to deviate from the linear line for small $v_b - v_0$.

We may summarise the results obtained in this work as follows. Depending on the probability p, the ratio of two resistances r_b/r_a, and the width w, two intrinsically different breaking behaviours can appear. For one case, breaking occurs only on the weakest bundle. For the other case, breaking propagates across bundles. Using simple probability theory, we derived that the configuration average voltage, v_b, required to break the system must increase by $\sim w^{-1/2}\exp(-w)$ in breaking propagation, and this increment turns out to be a more intrinsic quantity in the breaking process. The failure probability is of the second type of Weibull distribution with a non-universal Weibull constant. The Weibull constant depends on the system disorder, probability p and resistances r_a, r_b. Numerical simulation confirms the theoretical prediction. We also found that the total number of bonds broken is maximised by choosing the ratio of two resistances to be ~4 for $w \to \infty$. Finally we discussed the problem of a stand-by systems without repair in connection with the rupture of two-component fibrous bundles.

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